Crack-Tip Plastic Zones in Glassy Polymers under Small-Scale Yielding

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Synopsis

Two pressure-modified von Mises yield criteria were used for the determination of the plastic zones developed around cracks subjected to opening-mode loading conditions in glassy polymers. These criteria take into account the particular behavior of glassy polymers expressed by the dependence of their yield locus on the hydrostatic stress component and the difference in their tensile and compressive yield stresses. It was found that both criteria predict larger plastic zones than those obtained by the von Mises criterion which increase as the ratio of the compressive to tensile yield stress also increases. Furthermore, it was established that the differences in the predictions of plastic zones between the two criteria and the von Mises criterion increase with the Poisson's ratio of the material of the cracked plate.

INTRODUCTION

Linear elastic fracture mechanics based on the concept of stress intensity factor has successfully been used so far in predicting the failure behavior of engineering components. Application of this theory is justified if the yield zone accompanying the crack tip is smaller than the dimensions of the existing crack. The extend of the crack-tip yield zone influences the crack propagation velocities since these velocities are normally much less for plastic than for elastic strains. Thus, the determination of the yield zones existing at the tips of cracks is quite important.

Since the pioneering work of McClintock and Irwin,[1] many attempts[2–7] have been made to determine the size and shape of the plastic zone at the crack tip. All these studies were based on the von Mises or Tresca yield criteria, which have been proved successful for the description of plastic deformation in metals.

Most polymeric materials, however, do not obey such criteria, which are based on the assumptions of the independence of the yield behavior on the hydrostatic component of the stress state and the equality of the yield strengths in tension and compression. Experimental evidence in polymers showed that these hypotheses are critically questioned. Therefore, efforts have been made to establish yield criteria that adequately describe the behavior of polymers under multiaxial states of stress.

Whitney and Andrews[8] performed experiments on polystyrene, poly(methyl methacrylate), polycarbonate, and poly(vinyl chloride) under complex stress states and observed that their results do not fit either the von Mises or the Tresca criterion. They came to the conclusion that the yield behavior of these glassy polymers is affected by the hydrostatic component of stress and pointed out that the Coulomb criterion provides a better fit. The use of this criterion was also suggested by Bowden and Jukes[9] and Sternstein and Ongchin.[10] Bauwens,[11,12]

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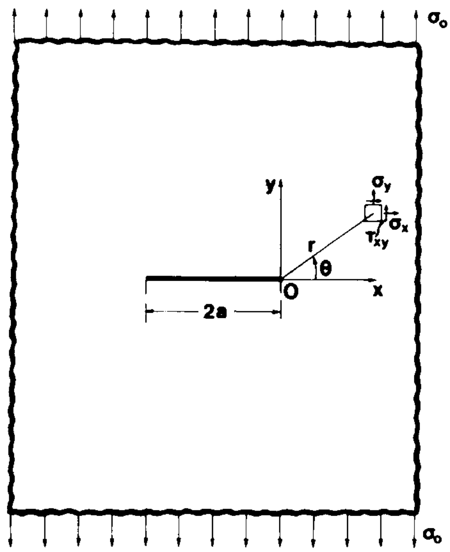

Fig. 1. Geometry of a cracked plate subjected to uniform uniaxial tensile stress σ_0 perpendicular to the axis of the crack.

using the Eyring theory of non-Newtonian flow, derived a yield condition that takes into account the influence of the hydrostatic stress and therefore disproves the von Mises yield criterion. Experimental data on poly(vinyl chloride) confirmed the proposed criterion. Christiansen et al.[13] used modified von Mises and Coulomb yield criteria to take into consideration the pressure dependence of yield behavior of glassy polymers. Sauer et al.[14] studied the effects of hydrostatic pressure up to 100,000 psi on the stress–strain behavior of poly(tetrafluoroethylene), which is a highly crystalline polymer, and polycarbonate, which is an amorphous polymer; they found an increase in the yield stress with increase in pressure. Raghava and Caddell[15] performed yield experiments in uniaxial tension and compression and biaxial stress states on high-density polyethylene. The experimental results were favorably compared with a pressure-modified von Mises criterion which takes into account the differences in tensile and compressive yield strengths as well as the hydrostatic component of the applied stress state. In another article, Raghava, Caddell, and Yeh[16] compared this criterion with the modified octahedral shear stress criterion and found that the predictions of the former are a little more reasonable. A review of the yield criteria and the work involving the yield behavior of polymers was made by Ward[17] in an interesting article.

The peculiar character of glassy polymers and their inadequacy to simulate the mechanical behavior of metals for photoplastic stress analyses was indicated by Theocaris[18] in an illuminating report. Theocaris discussed thoroughly the differences existing between glassy polymers and metals with respect to the stress–strain curve in tension, the viscoelastic behavior, the Poisson's ratio, and particularly the yield locus under multiaxial stresses. He showed that qualitative and quantitative differences exist between the mechanisms of plastic deformation in metals and in glassy polymers and concluded that the use of these materials for studying the plastic behavior of metals gives only qualitative results.

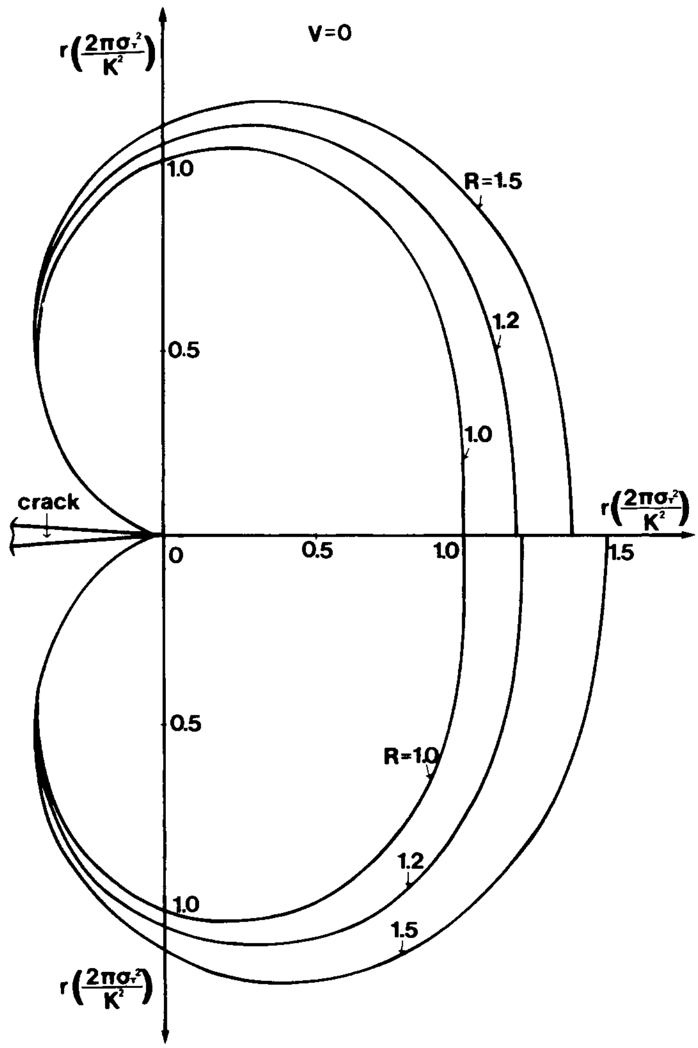

Fig. 2. Elastic–plastic boundary around the tip of a crack in a plate subjected to plane stress conditions ($\nu = 0$) for $R = 1.0, 1.2$, and 1.5. Curves of the upper half of the figure correspond to the PMOSC criterion, while curves of the lower half correspond to the PMMC criterion.

From the above brief review, it can be concluded that in studies concerning the yield behavior of glassy polymers under multiaxial stress states the characteristic behavior of these materials should be taken into account. Thus, yield criteria taking into consideration the dependence of yield behavior on the hydrostatic component of the stress state and the different values of the yield strengths in tension and compression should be used.

It is the purpose of the present report to study the shape and size of plastic zones developed around crack tips in glassy polymers using yield theories which adequately describe the particular characteristic behavior of these materials. The case of a crack in an infinite isotropic elastic plate loaded in tension perpendicularly to the crack axis is considered. The analysis is undertaken by using

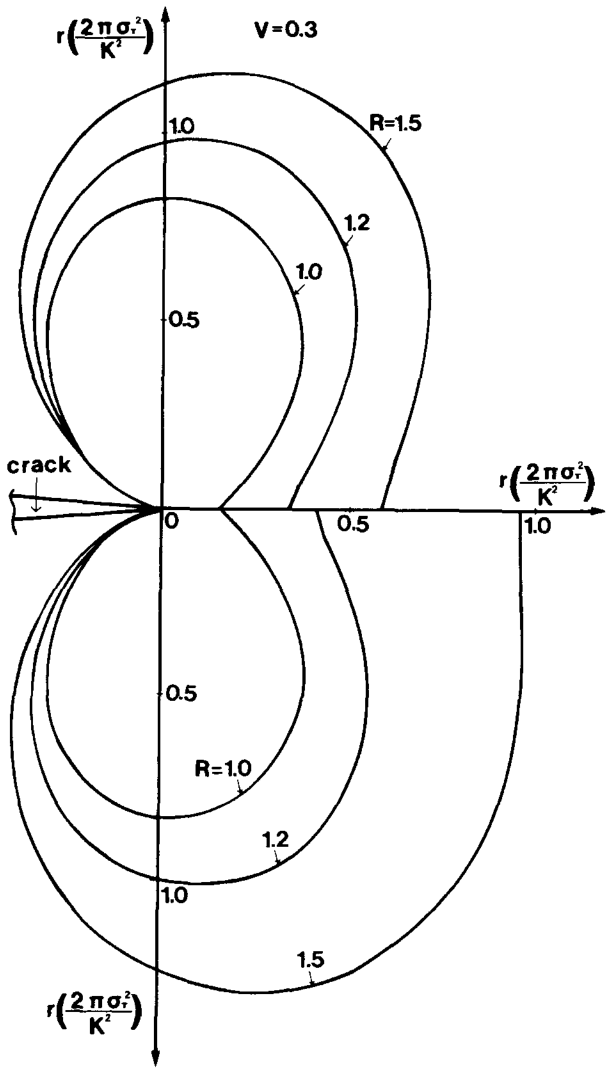

Fig. 3. As in Fig. 2 for a cracked plate subjected to plane strain conditions with $\nu = 0.3$.

two modified von Mises yield criteria. A detailed analysis of the plastic zones determined by these criteria incorporating the dependence of their geometric elements on the ratio of the compressive to tensile yield stress and the Poisson's ratio of the glassy polymer was undertaken.

Crack-Tip Stress Field

Consider a crack of length $2a$ in an infinite isotropic elastic plate subjected to a uniform tensile stress σ_0 perpendicular to the axis of the crack (Fig. 1). For this case, the singular stress field in the vicinity of the crack tip is given by the

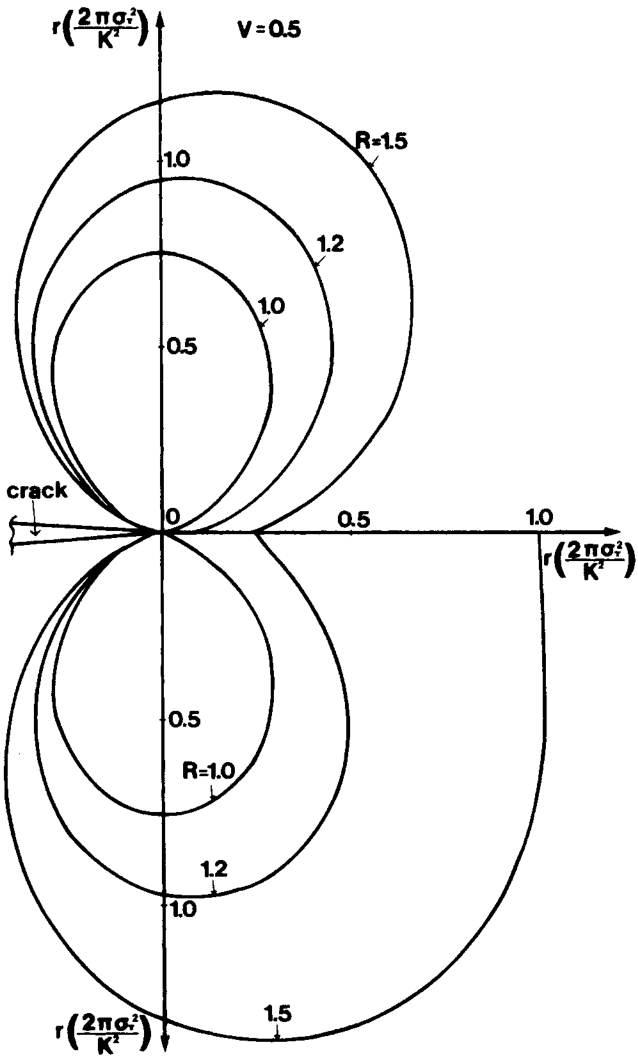

Fig. 4. As in Fig. 2 for a cracked plate subjected to plane strain conditions with $\nu = 0.5$.

following relations[19]:

$$\sigma_x = \frac{K}{(2\pi r)^{1/2}} \cos\frac{\theta}{2}\left(1 - \sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)$$

$$\sigma_y = \frac{K}{(2\pi r)^{1/2}} \cos\frac{\theta}{2}\left(1 + \sin\frac{\theta}{2}\cos\frac{\theta}{2}\right) \qquad (1)$$

$$\tau_{xy} = \frac{K}{(2\pi r)^{1/2}} \sin\frac{\theta}{2}\cos\frac{\theta}{2}\cos\frac{3\theta}{2}$$

where K is the so-called stress intensity factor which for the case of the infinite plate is equal to $\sigma_0\sqrt{\pi a}$. Relations (1) can be used to describe the stress field around crack tips in any symmetrical stress field independent of the geometric configuration of the plate and the corresponding loading conditions. In this case, however, K is equal to $k\sigma_0\sqrt{\pi a}$, where the correction factor k takes into account all the particular characteristics of the stress field under consideration.

TABLE I
Values of Normalized Radius r Along Various Polar Directions of the Elastic–Plastic Boundary at the Tip of a Crack in a Plate Subjected to Plane Stress Conditions for R = 1.0, 1.1, 1.2, 1.3, 1.4, and 1.5[a]

R	$r\ (2\pi\sigma_Y^2/K^2)$											
	$\theta = 0°$		30°		60°		90°		120°		150°	
1.0	1.000	1.000	1.120	1.120	1.312	1.312	1.250	1.250	0.812	0.812	0.254	0.254
1.1	1.092	1.099	1.206	1.211	1.374	1.378	1.280	1.281	0.820	0.821	0.255	0.255
1.2	1.173	1.198	1.279	1.301	1.426	1.440	1.305	1.311	0.827	0.828	0.255	0.255
1.3	1.244	1.294	1.343	1.386	1.472	1.498	1.327	1.339	0.833	0.836	0.256	0.256
1.4	1.306	1.386	1.400	1.469	1.511	1.554	1.346	1.364	0.838	0.842	0.256	0.256
1.5	1.361	1.476	1.449	1.548	1.545	1.606	1.362	1.388	0.842	0.848	0.256	0.257

[a] The values of the first and second columns correspond to PMOSC and PMMC criteria, respectively.

Using relations (1), the following values of the principal stresses $\sigma_{1,2}$ are obtained:

$$\sigma_{1,2} = \frac{K}{(2\pi r)^{1/2}} \cos\frac{\theta}{2}\left(1 \pm \sin\frac{\theta}{2}\right) \tag{2}$$

where the plus and minus signs correspond to σ_1 and σ_2, respectively.

The principal stresses of relations (2) correspond to the case of plane stress characterized by the zeroing of the third principal stress σ_3 normal to the plane of the plate. This case prevails in thin specimens and not in the very close vicinity of the crack tip. However, for thick specimens close to the crack tip, plane strain conditions characterized by the zeroing of the principal strain ϵ_3 dominate. In this case, σ_3 is not equal to zero but it is given by

$$\sigma_3 = \frac{2\nu K}{(2\pi r)^{1/2}} \cos\frac{\theta}{2} \tag{3}$$

TABLE II
As in Table I for Plane Strain Conditions and $\nu = 0.3$

R	$r\ (2\pi\sigma_Y^2/K^2)$											
	$\theta = 0°$		30°		60°		90°		120°		150°	
1.0	0.160	0.160	0.336	0.336	0.682	0.682	0.830	0.830	0.602	0.602	0.198	0.198
1.1	0.250	0.267	0.446	0.461	0.793	0.803	0.908	0.914	0.640	0.642	0.207	0.208
1.2	0.340	0.411	0.549	0.606	0.892	0.930	0.976	0.997	0.672	0.680	0.215	0.217
1.3	0.427	0.580	0.645	0.766	0.981	1.059	1.036	1.079	0.699	0.717	0.222	0.226
1.4	0.510	0.763	0.733	0.934	1.060	1.118	1.088	1.158	0.724	0.752	0.228	0.234
1.5	0.587	0.952	0.813	1.104	1.131	1.316	1.135	1.234	0.745	0.785	0.233	0.242

TABLE III
As in Table I for Plane Strain Conditions and $\nu = 0.5$

R	$r\ (2\pi\sigma_Y^2/K^2)$											
	$\theta = 0°$		30°		60°		90°		120°		150°	
1.0	0.000	0.000	0.187	0.187	0.562	0.562	0.750	0.750	0.562	0.562	0.187	0.187
1.1	0.018	0.074	0.297	0.319	0.695	0.710	0.852	0.860	0.614	0.618	0.201	0.202
1.2	0.062	0.250	0.407	0.496	0.817	0.873	0.942	0.973	0.660	0.674	0.213	0.216
1.3	0.119	0.479	0.514	0.706	0.927	1.045	1.021	1.087	0.699	0.728	0.223	0.230
1.4	0.183	0.734	0.616	0.934	1.028	1.222	1.092	1.199	0.734	0.780	0.232	0.243
1.5	0.250	1.000	0.712	1.169	1.119	1.399	1.156	1.309	0.765	0.830	0.240	0.255

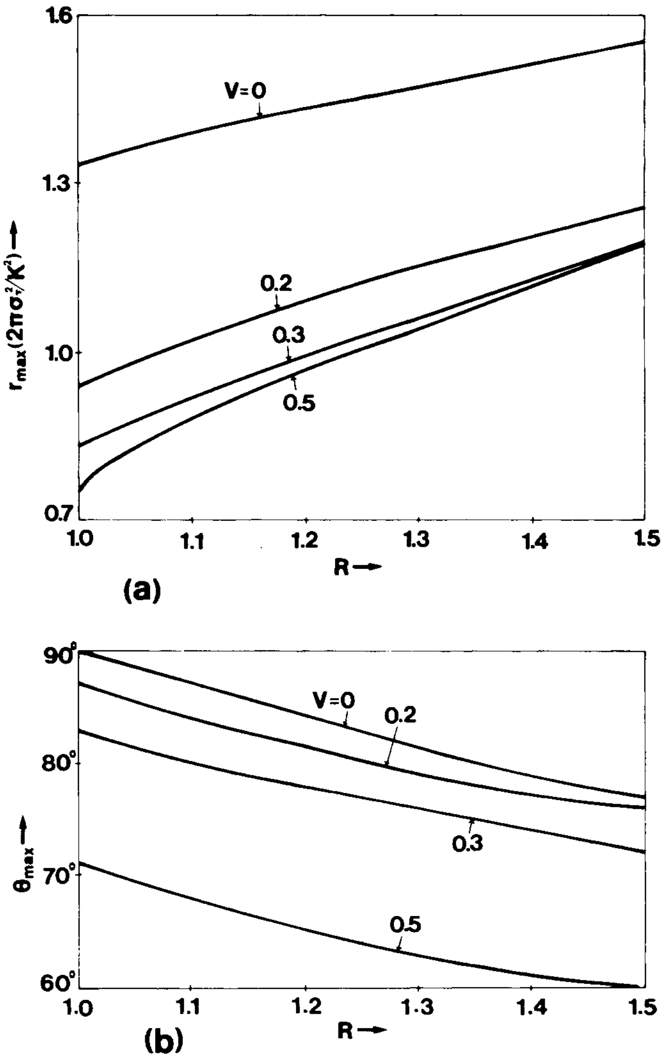

Fig. 5. Variation of the normalized maximum radius r_{max} of the elastic–plastic boundary around the tip of a crack subjected to plane strain conditions vs. R for $\nu = 0, 0.2, 0.3$, and 0.5 according to the PMOSC criterion (a). Values of the polar angle θ_{max} corresponding to r_{max} (b).

where ν is the Poisson's ratio of the material of the plate.

Fracture Criteria in Glassy Polymers

The more widely used fracture criterion for the description of the yield behavior under a multiaxial state of stress is the von Mises criterion, according to which yielding depends only on the deviatoric stress tensor and is independent of the hydrostatic part of the stress. This condition is expressed mathematically by the following relation:

$$\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - (\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1) = \sigma_y^2 \quad (4)$$

where σ_y is the yield strength of the material either in tension or in compression.

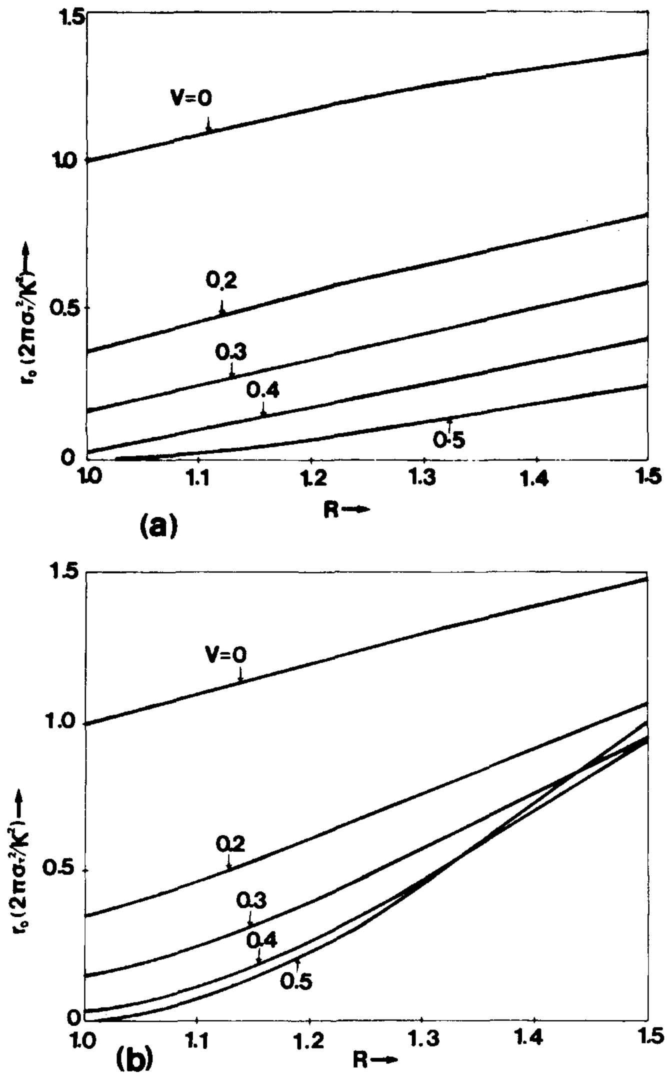

Fig. 6. As in Fig. 5(a) for the radius r_0 of the elastic–plastic boundary along the crack axis according to the PMOSC criterion (a). As in Fig. 6(a) according to the PMMC criterion (b).

The von Mises yield criterion has been thoroughly verified and extensively used in metals which obey its assumptions. However, in the case of glassy polymers whose yielding is dependent on the hydrostatic stress component and which present different yield strengths in tension and compression, the von Mises criterion cannot adequately be used to describe their yield behavior. Two main modifications to this criterion for glassy polymers have been introduced. The first was initially proposed by Nadai[20] and applied to the case of glassy polymers by Bauwens[12] and Sternstein and Ongchin.[10] This criterion is expressed in the following form:

$$(\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - \sigma_1\sigma_2 - \sigma_2\sigma_3 - \sigma_3\sigma_1)^{1/2} + \frac{R-1}{R+1}(\sigma_1 + \sigma_2 + \sigma_3) = \frac{2R}{R+1}\sigma_T \quad (5)$$

where

$$R = \sigma_c/\sigma_T \tag{6}$$

with σ_c and σ_T being the yield strengths in compression and tension, respectively. In the sequel, we will refer to this criterion as the pressure-modified octahedral shear stress criterion (PMOSC).

The second modified von Mises yield criterion was introduced by Raghava, Caddell, and Yeh[15,16] and is expressed by the following relation:

$$(\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - \sigma_1\sigma_2 - \sigma_2\sigma_3 - \sigma_3\sigma_1) + (\sigma_c - \sigma_T)(\sigma_1 + \sigma_2 + \sigma_3) = \sigma_c\sigma_T \tag{7}$$

In the sequel, this criterion will be called the pressure-modified von Mises criterion (PMMC).

It can be observed that both yield criteria are pressure dependent and take into account the differences in the yield strengths for tension and compression. By equating these stresses ($\sigma_c = \sigma_T$), both yield criteria expressed by relations (5) and (7) degenerate to the von Mises criterion expressed by relation (4).

In the following, both these yield criteria are used for the determination of the crack-tip plastic zones under opening-mode loading for small-scale yielding.

Crack-Tip Plastic Zones

It was previously shown that biaxial or triaxial states of stress for the cases of plane stress or plane strain prevail in the vicinity of the crack tip. The plastic zone surrounding the crack tip is separated from the remaining elastic material by the elastic–plastic boundary. For the determination of this boundary, the singular stress field solution expressed for the principal stresses from relations (2) and (3) is used. Introducing these values of the principal stresses into the appropriate yield criterion, an equation describing the elastic-plastic boundary is obtained. Thus, for the case of the von Mises yield criterion, the following equations in polar form are obtained:

$$r\left(\frac{2\pi\sigma_Y^2}{K^2}\right) = \left(1 + 3\sin^2\frac{\theta}{2}\right)\cos^2\frac{\theta}{2} \tag{8}$$

for plane stress and

$$r\left(\frac{2\pi\sigma_Y^2}{K^2}\right) = \frac{1}{4}\,[2(1-2\nu)^2 + 3(1-\cos\theta)](1+\cos\theta) \tag{9}$$

for plane strain conditions.

For the case of the PMOSC criterion expressed by relation (5), we obtain

$$r\left(\frac{2\pi\sigma_T^2}{K^2}\right) = \left(\frac{R+1}{2R}\right)^2\left[\left(1 + 3\sin^2\frac{\theta}{2}\right)^{1/2} + 2\left(\frac{R-1}{R+1}\right)\right]^2\cos^2\frac{\theta}{2} \tag{10}$$

for plane stress and

$$r\left(\frac{2\pi\sigma_T^2}{K^2}\right) = \left(\frac{R+1}{2R}\right)^2\left\{\frac{1}{2}\,[[2(1-2\nu)^2 + 3(1-\cos\theta)](1+\cos\theta)]^{1/2} + 2(1+\nu)\left(\frac{R-1}{R+1}\right)\cos\frac{\theta}{2}\right\}^2 \tag{11}$$

for plane strain.

Finally, for the case of the PMMC criterion expressed by relation (7), we obtain

$$r\left(\frac{2\pi\sigma_T^2}{K^2}\right) = \frac{1}{R^2}\left\{(R-1)\cos\frac{\theta}{2} + \left[(R-1)^2\cos^2\frac{\theta}{2} + R\cos^2\frac{\theta}{2}\left(1 + 3\sin^2\frac{\theta}{2}\right)\right]^{1/2}\right\}^2 \tag{12}$$

for plane stress and

$$r\left(\frac{2\pi\sigma_T^2}{K^2}\right) = \frac{1}{4R^2}\left\{2(1+\nu)(R-1)\cos\frac{\theta}{2} + \left\{4(1+\nu)^2(R-1)^2\cos^2\frac{\theta}{2} + R(1+\cos\theta)[2(1-2\nu)^2 + 3(1-\cos\theta)]\right\}^{1/2}\right\}^2 \tag{13}$$

for plane strain.

It can be observed that formulas (10) and (12) for plane stress conditions and formulas (11) and (13) for plane strain conditions coincide with formulas (8) and (9), respectively, for $R = 1$ (equal yield strengths in tension and compression). Furthermore, relations for plane strain coincide with the corresponding relations for plane stress when $\nu = 0$. Thus for $\nu = 0$, relations (9), (11), and (13) coincide with relations (8), (10), and (12), respectively.

In the following, the above developed formulas (8) to (13) will be used for the determination of the characteristic features of the crack-tip plastic zones under opening-mode loading conditions in glassy polymers.

Results

Figures 2, 3, and 4 present the elastic–plastic boundary around the crack tip under plane strain conditions for the values of Poisson's ratio ν of the material of the plate equal to 0, 0.3, and 0.5, respectively. The curves of the figures correspond to the values of the constant $R = 1.0$, 1.2, and 1.5. The elastic–plastic boundaries of the upper halves of the figures were drawn according to the PMOSC criterion, the elastic–plastic boundaries of the lower halves according to the PMMC criterion. The curves for $R = 1$ correspond to the von Mises yield criterion. It can be observed from these figures that the plastic zone surrounding the crack tip increases as the ratio R of the compressive σ_c to tensile σ_T strength of the material also increases. It was found that in the literature $R = 1.30$ for PVC,[16,12] PS,[12] and PMMA[10] and $R = 1.20$ for PC.[16] Therefore, the use of pressure-dependent criteria for the prediction of crack-tip plastic zones in these polymers is quite necessary. Furthermore, by comparing the predictions of the two criteria we observe that the PMMC criterion yields larger plastic zones than the PMOSC criterion.

In order to have a better picture of the results of the two criteria used for the determination of the elastic–plastic boundaries around crack tips, the radii of the plastic zones for the angles $\theta = 0$, 30, 60, 90, 120, and 150° and the values of $R = 1.0$, 1.1, 1.2, 1.3, 1.4, and 1.5 are shown in Tables I, II, and III for $\nu = 0$, 0.3, and 0.5, respectively. From these tables it can be observed that the difference in the plastic zone size according to both criteria for $R = 1$ and $R > 1$ is greater in the interval $0° < \theta < 90°$ than in the interval $90° < \theta < 180°$. We also observe that the effect of R on the plastic zone size is more pronounced for large values of the Poisson's ratio ν. We also see that as ν increases, the deviation of the predictions of the two yield criteria also increases. Furthermore, this deviation increases with R. The difference in the predictions of yielding of polymers between the PMMC and PMOSC criteria was also indicated by Raghava et al.[16] who pointed out that this difference increases with R. As was established by Raghava et al.,[16] the PMMC criterion more properly reflects the differences in the increase of yield strength under increasing hydrostatic pressure than the PMOSC criterion does. Therefore, the results concerning the plastic zones around cracks based on the PMMC criterion should be considered closer to the reality than those based on the PMOSC criterion.

The variation of the normalized maximum radius r_{max} of the elastic–plastic boundary versus R according to the PMOSC criterion is shown in Figure 5(a) for $\nu = 0, 0.2, 0.3$, and 0.5. We observe that r_{max} increases with R and that the difference in the values of r_{max} for $R = 1$ and $R > 1$ becomes more pronounced as the Poisson's ratio ν increases. The values θ_{max} of the polar angle θ for which the maxima in r occur are shown in Figure 5(b). Finally, Figures 6(a) and 6(b) present the variation of the extend of the plastic zone along the crack line $[r_0(2\pi\sigma_T^2/K^2)]$ versus R for $\nu = 0, 0.2, 0.3, 0.4$, and 0.5 according to the PMOSC and PMMC criteria, respectively.

CONCLUSIONS

In the present article, a thorough analysis of the plastic zones developed around the tips of a crack under opening-mode loading conditions in glassy polymers was undertaken. Two pressure-modified von Mises yield criteria which take into account the dependence of the yield behavior of glassy polymers on the hydrostatic stress component and the difference of their tensile and compressive yield stresses were used. From the whole study, the following conclusions may be derived:

(1) The crack-tip plastic zones calculated by the pressure-modified criteria are quite different than those determined by the usual von Mises criterion. This indicates the necessity of using these criteria in the determination of the plastic zones.

(2) The thus calculated crack-tip plastic zones by the pressure-modified criteria are always larger than those obtained by the von Mises criterion.

(3) The difference of the size of plastic zones determined by the pressure-modified and the von Mises criteria increases as the ratio of the compressive to tensile yield stress of the material increases. Furthermore, this difference increases as the value of the Poisson's ratio also increases.

(4) The difference of the size of plastic zones determined by the pressure-modified and the von Mises criteria is more pronounced for small values of the polar angle, and it decreases as this angle increases.

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